Philosophy of Quantum Mechanics

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1. (a) State the von Neumann "no hidden variables" theorem. What is Bell's objection to it?

In 1932 von Neumann published a proof that dispersion free[1] quantum states are impossible. He concluded from this that hidden variable theories are impossible. Von Neumann's proof of the impossibility of dispersion free states depends crucially on a particular assumption. Let A be some quantum mechanical observable and so $\langle\psi|A|\psi\rangle$ is the expectation value of an ensemble of particles prepared in the state $|\psi\rangle$. Now, for considerations of hidden variables, the individual members of that ensemble have definite values of observables. Let $\nu(A)$ denote the value of A taken by some particular particle of the ensemble. Von Neumann's assumption was that if A, B and C are *any* observables such that $C = A + B$, then the value of C assigned to a particle must satisfy

$$\nu(C) = \nu(A) + \nu(B). \tag{1}$$

It turns out that a theory that satisfies (1) cannot match the predictions of quantum mechanics, which is what drives von Neumann's proof. Bell objects very strongly to von Neumann's assumption, an assumption that Bell considered silly. It is silly precisely because there is no reason why (1) should hold if A and B do not commute. (1) does hold for commuting observables and, indeed, in general for expectation values, but if A and B do not commute then they do not have simultaneous eigenvalues and so they cannot be simultaneously measured. Thus there is no reason why (1) should hold for individual members of an ensemble and so no reason why (1) should be required by a hidden variables theory.

1. (b) State and prove the Bell-Kochen-Specker theorem. Why does Bell, unlike Kochen and Specker, not regard this as a no-hidden variables proof?

BKS Theorem: Let $\mathcal{H}$ be a Hilbert space of QM state vectors of dimension $D \geq 3$. Let M be a set containing n observables, defined by operators on $\mathcal{H}$. Then, for specific values of D and n, the following two assumptions are contradictory:
(BKS1) All n members of M simultaneously have values, i.e. are unambiguously mapped

[1]A state is considered dispersion free if for some observable A $\langle(A - \langle A\rangle)^2\rangle = \langle A^2\rangle - \langle A\rangle^2 = 0$. So a dispersion free state has an exact value or, stated otherwise, has no statistical character.

onto real unique numbers (designated, for observables A, B, C, ... by $\nu(A)$, $\nu(B)$, $\nu(C)$, ...).
(BKS2) Values of observables conform to the following constraints:

1. If A, B, C are all compatible and $C = A + B$, then $\nu(C) = \nu(A) + \nu(B)$;
2. if A, B, C are all compatible and $C = AB$, then $\nu(C) = \nu(A)\nu(B)$.

I will provide Mermin's proof of the BKS theorem in 4 dimensions. To prove the theorem we must demonstrate a set of observables $A, B, C, \ldots$ for which we can show that it is impossible to associate with each observable one of its eigenvalues $\nu(A), \nu(B), \nu(C), \ldots$ in such a way that (BKS2) is satisfied. Exploiting the properties of four dimensional Hilbert spaces, we will represent observables in terms of the Pauli matricies for two independent spin-1/2 particles σ^1_μ and σ^2_ν.[2] Now, consider the nine observables

$$\begin{array}{ccc} \sigma^1_x \otimes I & I \otimes \sigma^2_x & \sigma^1_x \otimes \sigma^2_x \\ I \otimes \sigma^2_y & \sigma^1_y \otimes I & \sigma^1_y \otimes \sigma^2_y \\ \sigma^1_x \otimes \sigma^2_y & \sigma^2_x \otimes \sigma^1_y & \sigma^1_z \otimes \sigma^2_z \end{array}$$

Now we need to show that it is impossible to assign values to all nine observables. To do this we make the following observations: (1) The observables in each of the three rows and each of the three columns mutually commute. This is clear from the fact that $(\sigma^i_j)^2 = 1$ for $i = 1, 2, j = x, y$ and the fact that σ^i_j and σ^i_k anticommute for $i = 1, 2$ and $j, k \in \{x, y, z\}, i \neq j$. (2) The product of the three operators in the right most column is $-I \otimes I$ and the product of the three operators in the other two columns and all three rows is $I \otimes I$, which follows from the facts that $\sigma^i_x \sigma^i_y = i\sigma^i_z$ for $i = 1, 2$ and $(\sigma^i_j)^2 = 1$ for $i = 1, 2, j = x, y, z$. (3) By (BKS2), the identities in (2) require the product of the values assigned to the observables in each row to be 1 and the product fo each of the observables in each column to be 1 for the first two columns and -1 for the last column. (4) The proof comes from the fact that an attempt to satisfy (3) leads to contradiction since the row identities require the product of all nine values to be 1 and the column identities require the same product to be -1. QED.

Bell does not regard this as a no hidden-variables proof *tout court* since, as he points out immediately after his proof, we have "tacitly assumed that the measurement of an observable must yield the same value independently of what other measurements must be made simulaneously." We assumed that the values taken by each of the observables do not depend on what measurements are being made on the system. Since the observables in the two sets do not mutually commute, there is no reason why we should make this requirement. Thus, we have assumed that the values taken by observables are noncontextual. So, what the BKS theorem shows is that hidden-variables theories must be contextual.

[2]These observables have the following useful properties: they square to 1 and so their eigenvalues are ± 1; each component of σ^1_μ commutes with each component of σ^2_ν; when μ and ν specify orthogonal directions, σ^i_μ anticommutes with σ^i_ν for $i = 1, 2$; and $\sigma^i_x \sigma^i_y = i\sigma^i_z$ for $i = 1, 2$.

2. State Bell's theorem. Explain the conditions that Shimony calls "parameter independence" and "outcome independence." What is the relation between these conditions and the Bell locality (factorizability) condition? What is the relevance of this distinction to the issue of determinism?

To state Bell's theorem we will consider a pair of spin-1/2 particles in the spin singlet state separated by some distance. Let $p_\lambda(x_1, y_2|a, b)$ be the probability that, if spin experiments are performed in the a-direction of particle 1 and the b-direction on particle 2, the results will be x and y for the respective particles. Let $P(x_1, y_2|a, b) = \langle p_\lambda(x_1, y_2|a, b)\rangle$ be the ensemble expecation values where $\langle\rangle$ denotes averaging of the λ, *i.e.* hidden-variable, distribution. Now, the crucial assumption required for the theorem is the *Bell factorizability condition* (BF):

$$p_\lambda(x_1, y_2|a_1, b_2) = p_\lambda^{(1)}(x_1|a_1)p_\lambda^{(2)}(y_2|b_2)$$

for some probability functions $p_\lambda^{(1)}(x_1|a_1)$ not dependent on y_2 and b_2, and $p_\lambda^{(2)}(y_2|b_2)$, not dependent on x_1 and a_1. Now we are in a position to state Bell's theorem. Bell's Theorem: If the microscopic distributions $p_\lambda^{(1)}(x_1, y_2|a_1, b_2)$ satisfy the Bell factorizability condition, then the ensemble values satisfy

$$0 \leq P(+,-|a,b') + P(-,+|a',b) + P(+,+|a',b') - P(+,+|a,b) \leq 1.$$

Quantum mechanics predicts that this inequality will be violated and experiment has fallen on the side of quantum mechanics. Thus any hidden-variables theory must reject one of the assumptions behind the theorem, which is the Bell factorizability criterion.

Shimony's *parameter independence condition* (PI) is the condition that

$$p_\lambda^{(1)}(x_1|a, b) = p_\lambda^{(1)}(x_1|a, b')$$

and

$$p_\lambda^{(2)}(y_2|a, b) = p_\lambda^{(2)}(y_2|a', b).$$

This amounts to the condition that the direction of measurement on the distant particle does not affect the measurements on the local particle. Shimony's *outcome independence condition* (OI) is the condition that

$$p_\lambda(x_1, y_2|a, b) = p_\lambda^{(1)}(x_1|a, b)p_\lambda^{(2)}(y_2|a, b).$$

This amounts to the condition that the measurement outcomes for the two particles are independent. Jarret made the interesting observation that

$$\text{BF} \equiv \text{PI} \wedge \text{OI}.$$

Now, since experiment forces us to reject (BF), this implies that we must reject (PI) or (OI). A deterministic theory must satisfy the (OI) criterion. This is so because determinism forces the outcomes of any experiment to be already determined by specification of initial conditions and so the outcome of the measurement on one particle cannot depend on the outcome of measurement of the other. Thus, a deterministic theory is forced to reject (PI) which implies that deterministic hidden-variables theories must be non-local. The de-Broglie-Bohm theory is such a hidden variables theory.

3. What is the "measurement problem" in quantum mechanics? Briefly outline three major approaches towards solving (or dissolving) it.

In quantum mechanics observables are represented by hermitian operators on a complex Hilbert space $\mathcal{H}$. Quantum states are represented by vectors in $\mathcal{H}$, the state vector for the system. For any observable A, the possible observable values are its eigenvalues and the states corresponding to each observed value are the eigenvectors corresponding to the eigenvalue. A given state vector is, in general, in a superposition of observable states (eigenvectors) for a given observable. The evolution of this state vector is then correctly described by the action of some unitary operator and, in the case of time evolution, described by the Schrödinger equation. Now, in the case of measurement interactions, the system ends up in an eigenstate of the observable in question, which is a non-unitary (non probability conserving) process and so is not described by the quantum dynamics. Quantum mechanics correctly predicts the probability of measuring certain values for the system, *i.e.* it yields the correct statistical averages for measurements on ensembles of particles in the same state, but cannot explain how the reduction of a superposition into an eigenstate takes place. The solution to (or dissolution of) the problem of how this reduction takes place is what is known as the measurement problem in quantum mechanics.

I will now briefly outline three of the major approaches that have been proposed to solve the measurement problem. The first such proposal is the de Broglie-Bohm hidden variables interpretation of quantum mechanics. In this interpretation, the particles of a system always have a well-defined position and this position determines all other measurable properties of the system. The evolution of the system is effected by a guiding wave, or pilot wave, that is always described by the Schrödinger equation. Since particles always have a well-defined position, and the other properties are determined by this,[3] no state vector reduction ever takes place, which is why it is a solution to the measurement problem. Another proposal is the continuous spontaneous localization (CSL) model. This approach actually modifies the Schrödinger equation by adding a stochastic term that preserves neither unitarity or linearity. This modifyies the quantum dynamics in such a way as to solve the measurement problem. The solution comes from the fact that the additional term mediates interaction with an all pervading non-local noise field that functions to drive superpositions of states into a particular eigenstate. Thus, the dynamics is expanded to explain the reduciton process. Such dynamical redction only happens for states that differ in terms of their centre of mass (COM) and so the theory singles out a special basis, namely the position basis. The theory has two adjustable constants which determine the minimum different in COM required to trigger reduction and another constant that determines the time scale of the reduction process. A third proposal to solve the measurement problem is to take an instrumentalist view of quantum mechanics. On this view, variants of which are the Copenhagen and statistical interpretations, the state vector is not identified with an actual physical system, it is just a representation of our state of knowledge of some physical system. The formalism of quantum mechanics, then,

[3]There are issues with contextuality which can be explained by the interaction of the system with the measurement apparatus, but we need not consider this here.

just provides us with very accurate predictions of what we will observe given a particular experimental set up. This final solution is, thus, really a dissolution of the problem.

4. Briefly outline the EPR argument, with careful attention to the premises used.

Consider a pair of spin-1/2 particles prepared in the spin singlet state that are moving freely in opposite directions. Spin measurements on these particles are made using Stern-Gerlach magnets oriented in some direction. Let the operators corresponding to the measurements on the two particles be σ_1 and σ_2. Since the particles are in the spin singlet state, no matter what direction the SG magnets are oriented, so long as they are oriented in the same direction, if a measurement on one particle is $+1$, then the measurement on the other will be -1. Stated more precisely, if we let $\vec{a}$, where $|\vec{a}| = 1$ be the direciton of the inhomogeneity of the magnetic field of the SG magnets, then if $\sigma_1 \cdot \vec{a}$ yields the value 1, then $\sigma_2 \cdot \vec{a}$ will yield the value -1, and vice versa. We now introduce an assumption, namely that if the two measurements are made at places remote from one another the orientation of one magnet does not influence the result obtained with the other. This assumption assumes a form of locality, where we reject instantaneous action at a distance, and separability, which is the assumption that the two particles can be described independently. Due to the fact that we can predict with certainty the value of any component of σ_2 with a knowledge of the same component of σ_1, the result of any such measurement must already be determined. Then, since the initial quantum mechanical wavefunction does not determine the result of an individual measurement, this predetermination then implies the possibility of a more complete specification of the state of the system. With this EPR were led to the conclusion that the formalism of QM is not complete. This does not follow, however, since it possible to reject the assumption made by rejecting locality or separability or both. Indeed, it seems that the separability criterion fails due to the fact that the two particles are in an entangled state, *i.e.* there are no single particle bases that enable us to describe the two particle system as a product of two single particle state vectors.

5 (a). Consider the following state-vector of a system consiting of two spin-1/2 particles:

$$|\psi\rangle = \frac{1}{2}(|z+\rangle \otimes |z+\rangle - |z+\rangle \otimes |z-\rangle + |z-\rangle \otimes |z+\rangle - |z+\rangle \otimes |z-\rangle). \quad (2)$$

Is this an entangled state? Explain your answer.

The state $|\psi\rangle$ can be rewritten as

$$|\psi\rangle = \frac{1}{2}(|z+\rangle \otimes |z+\rangle - 2|z+\rangle \otimes |z-\rangle + |z-\rangle \otimes |z+\rangle). \quad (3)$$

In order to determine if $|\psi\rangle$ is entangled we need to see if it is possible to factorize the state into a product of single particle states. The form of the state description that we seek is

$$|\psi\rangle = (a_1|z+\rangle + b_1|z-\rangle) \otimes (a_2|z+\rangle + b_2|z-\rangle). \quad (4)$$

From (4) we obtain

$$|\psi\rangle = a_1a_2|z+\rangle \otimes |z+\rangle + a_1b_2|z+\rangle \otimes |z-\rangle + a_2b_1|z-\rangle \otimes |z+\rangle + b_1b_2|z+\rangle \otimes |z-\rangle. \quad (5)$$

Now, matching up terms of (5) with (3) we obtain $a_1a_2 = 1/2, a_1b_2 = -1, a_2b_1 = 1/2$ and $b_1b_2 = 0$. This last equation implies that $(b_1 = 0) \vee (b_2 = 0)$. If $b_1 = 0$ then this implies that $0 = a_2b_1 = 1/2$, which is a contradiction. On the other hand, if $b_2 = 0$, then this implies that $0 = a_1b_2 = -1$, which is a contradiction. Since either possibility leads to contradiction there are no values for a_1, a_2, b_1 and b_2 that satisfy (4), thus $|\psi\rangle$ is an entangled state. If the final term in (2) had been $-|z-\rangle \otimes |z-\rangle$ rather than $-|z+\rangle \otimes |z-\rangle$, then $|\psi\rangle$ would not have been entangled.

5 (b). Is the following state vector entangled? Explain your answer.

$$|\varphi\rangle = \frac{1}{\sqrt{3}}(|z+\rangle \otimes |z+\rangle - |z+\rangle \otimes |z-\rangle + |z-\rangle \otimes |z+\rangle). \quad (6)$$

Once again, the form of the state description that we seek is

$$|\varphi\rangle = (a_1|z+\rangle + b_1|z-\rangle) \otimes (a_2|z+\rangle + b_2|z-\rangle). \quad (7)$$

From (7) we obtain

$$|\varphi\rangle = a_1a_2|z+\rangle \otimes |z+\rangle + a_1b_2|z+\rangle \otimes |z-\rangle + a_2b_1|z-\rangle \otimes |z+\rangle + b_1b_2|z+\rangle \otimes |z-\rangle. \quad (8)$$

Now, matching up terms of (8) with (6) we obtain $a_1a_2 = 1/\sqrt{3}, a_1b_2 = -1/\sqrt{3}, a_2b_1 = 1/\sqrt{3}$ and $b_1b_2 = 0$. This last equation implies that $(b_1 = 0) \vee (b_2 = 0)$. If $b_1 = 0$ then this implies that $0 = a_2b_1 = 1/\sqrt{3}$, which is a contradiction. On the other hand, if $b_2 = 0$, then this implies that $0 = a_1b_2 = -1/\sqrt{3}$, which is a contradiction. Since either possibility leads to contradiction there are no values for a_1, a_2, b_1 and b_2 that satisfy (7), thus $|\varphi\rangle$ is an entangled state.